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A GENERALIZATION OF THE CONTOU-CARRÈRE SYMBOL

BY

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ABSTRACT

Using techniques of Algebraic Geometry, the aim of this paper is to give a generalized definition of the Contou-Carrère symbol as a morphism of schemes. In fact, from formal schemes and Heisenberg groups, we provide a new definition of the Contou-Carrère symbol and a generalization of it associated with a separable extension $k \hookrightarrow k(s)$. Moreover, a reciprocity law is proved and classical explicit reciprocity laws are recovered from it.

1. Introduction

Given a complete curve C over an algebraically closed field and a closed point $p \in C$, in 1959 J. P. Serre [14] defined the multiplicative local symbol as

$$(f,g)_p = (-1)^{n\cdot m}\frac{f^n}{g^m}(p) \quad \text{with } n = v_p(g), \quad m = v_p(f).$$

A few years later, in 1971, J. Milnor [10] defined the tame symbol d_v associated with a discrete valuation v on a field F. Explicitly, if A_v is the valuation ring, p_v is the unique maximal ideal and $k_v = A_v/p_v$ is the residue class field, Milnor defined $d_v \colon F^* \times F^* \to k_v^*$ by

$$d_v(x,y) = (-1)^{v(x)\cdot v(y)}\frac{x^{v(y)}}{y^{v(x)}}(\operatorname{mod} p_v).$$

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This definition generalizes Serre's definition of the multiplicative local symbol and J. Milnor proved that the tame symbol is a continuous Steinberg symbol. During the last 30 years, the tame symbol has been used in Algebraic K-Theory to study the group K_2F.

The analytic construction of the tame symbol was found in 1979 by P. Deligne as a morphism in the derived category of the category of sheaves of abelian groups over a Riemann surface, and also independently by A. Beilinson [4]. Deligne's article [8] was published in 1991. Moreover, in 1994 C. Contou-Carrère [7] defined a functor from the category of formal noetherian schemes to the category of groups, which allowed him to give a generalization of the tame symbol as a morphism of functors.

Recently, G. W. Anderson and the author have proved a reciprocity law for the Contou-Carrère symbol [1], using a similar method to Tate's proof of the residue theorem [15] and the proof of E. Arbarello, C. de Concini and V. G. Kac of the reciprocity law of the tame symbol for a complete curve [2].

Furthermore, A. Beilinson, S. Bloch and H. Esnault [6] have found a construction of the Contou-Carrère symbol as the commutator pairing in a Heisenberg super extension, and from this definition they have obtained another proof of the reciprocity law for this symbol.

Using techniques of Algebraic Geometry, the goal of this paper is to give a generalized definition of the Contou-Carrère symbol as a morphism of schemes. Indeed, we provide a new definition of the Contou-Carrère symbol and a generalization of it associated with a separable extension $k \hookrightarrow k(s)$. Our definition coincides with the expression given by C. Contou-Carrère by considering S-valued points, S being a connected k-scheme.

As far as we know our characterization of the Contou-Carrère symbol, defined from duality morphisms of groups schemes, and the provided generalization of it, including a generalized reciprocity law, are not stated explicitly in the literature.

The organization of the paper is as follows:

In Section 2, the formal k-scheme $\widetilde{\Gamma}$ is introduced, where k is an arbitrary field. This section also briefly introduces generalized Witt rings, in order to determine the duality between the group schemes Γ_+ and Γ_-.

Section 3 deals with the main results of this paper. Namely, we define a Heisenberg group, $\mathcal{H}(\widetilde{\Gamma})$, associated with the formal scheme $\widetilde{\Gamma}$ and give a definition of the Contou-Carrère symbol as a morphism of schemes from the commutator of the group extension induced by $\mathcal{H}(\widetilde{\Gamma})$. This morphism, $(\, , \,)_k \colon \widetilde{\Gamma} \times \widetilde{\Gamma} \to \mathbb{G}_m$, is characterized as a differentiated element in the cohomology class $[e_{\mathcal{H}(\widetilde{\Gamma})}] \in$

$H^2_{reg}(\widetilde{\Gamma}, \mathbb{G}_m)$, $e_{\mathcal{H}(\widetilde{\Gamma})}$ being the commutator referred to above. Using a similar method, we provide a generalization of this symbol associated with a separable extension $k \hookrightarrow k(s)$.

Finally, Section 4 is devoted to proving a reciprocity law for the generalized Contou-Carrère symbol defined in Section 3 that is valid for complete algebraic curves over a perfect field and to recovering classical explicit reciprocity laws from it.

For a detailed study of formal groups and Witt rings, the reader is referred to [9].

2. Preliminaries

2.A. FORMAL k-SCHEME $\widetilde{\Gamma}$. Let k be an arbitrary field. First, we shall recall the definition of the formal group Γ (for a complete study see [3]).

Γ is defined as the formal group scheme $\Gamma_- \times \mathbb{G}_m \times \Gamma_+$ over $\operatorname{Spec} k$, where Γ_- is the formal scheme representing the functor on groups:

$$S \rightsquigarrow \Gamma_-(S) = \left\{ \begin{array}{c} \text{series } a_n z^{-n} + \cdots + a_1 z^{-1} + 1 \\ \text{where } a_i \in H^0(S, \mathcal{O}_S) \text{ are} \\ \text{nilpotents and } n \text{ is arbitrary} \end{array} \right\},$$

$\mathbb{G}_m$ is the multiplicative group, and the scheme Γ_+ represents

$$S \rightsquigarrow \Gamma_+(S) = \left\{ \begin{array}{l} \text{series } 1 + a_1 z + a_2 z^2 + \cdots \\ \text{where } a_i \in H^0(S, \mathcal{O}_S) \end{array} \right\}.$$

The group laws of Γ_- and Γ_+ are those induced by the multiplication of series. Let us now denote by $\widetilde{\Gamma}$ the formal group scheme $\mathbb{Z}_* \times \Gamma$ over $\operatorname{Spec} k$, where $\mathbb{Z}_* = \coprod_{\alpha \in \mathbb{Z}} \operatorname{Spec} k$.

For each locally noetherian k-scheme S one has that

$$\mathbb{Z}_*^\bullet(S) = \operatorname{Map}_{cont.}(S, \mathbb{Z}),$$

considering $\mathbb{Z}$ as a discrete topological space. Moreover, the above equality determines a structure of group scheme on $\mathbb{Z}_*$, which coincides with the group structure of $\mathbb{Z}$ when we consider its rational points.

Furthermore, if S is a connected k-scheme, we have that

$$\widetilde{\Gamma}^\bullet(S) = (H^0(S, \mathcal{O}_S)[[z]][z^{-1}])^*,$$

and hence $\widetilde{\Gamma}^\bullet(\operatorname{Spec} k) = (k[[z]][z^{-1}])^* = (k[[z]])_0^*$.

2.B. Generalized Witt rings. Let us consider a vector with infinite variables, $x = (x_1, x_2, \ldots, x_n, \ldots)$, and, for each $n \in \mathbb{N}$, let $\omega_n(x)$ be the polynomial

$$\omega_n(x) = \sum_{d/n} d x_d^{n/d}.$$

Then, given the vectors $x = (x_i)$ and $y = (y_j)$, one can define the series of polynomials $\Sigma = (\Sigma_i(x,y))$ and $\Pi = (\Pi_j(x,y))$ by the equations

$$\omega_n(\Sigma) = \omega_n(x) + \omega_n(y) \quad \text{and} \quad \omega_n(\Pi) = \omega_n(x) \cdot \omega_n(y).$$

Definition 2.1: Given a ring A —commutative and with unit element— one defines a generalized Witt ring, $\mathbb{W}(A)$, to be the set of infinite sequences $a = (a_1, a_2, \ldots)$, with $a_i \in A$, together with the ring operations

$$(a_1, a_2, \ldots) + (b_1, b_2, \ldots) = (\Sigma_1(a,b), \Sigma_2(a.b), \ldots),$$
$$(a_1, a_2, \ldots) \cdot (b_1, b_2, \ldots) = (\Pi_1(a,b), \Pi_2(a,b), \ldots).$$

With this definition, $\mathbb{W}(A)$ is commutative and with unit element ring ([9], page 117).

We now denote by $\mathbb{W}_+(A)$ the abelian group induced by the structure of a generalized Witt ring.

Let $\xi_1, \xi_2, \ldots; \eta_1, \eta_2, \ldots$ be indeterminates. We define the sequences $\bar{x} = (\bar{x}_i)$ and $\bar{y} = (\bar{y}_j)$ by the equations

$$\prod_i (1 - \xi_i t) = 1 + \bar{x}_1 t + \bar{x}_2 t^2 + \cdots \quad \text{and} \quad \prod_i (1 - \eta_i t) = 1 + \bar{y}_1 t + \bar{y}_2 t^2 + \cdots.$$

Hence, from these definitions we can construct a polynomial sequence $P_1, P_2, \ldots$ satisfying the relations

$$\prod_{i,j} (1 - \xi_i \eta_j t) = 1 + P_1 t + P_2 t^2 + \cdots.$$

Bearing in mind the fundamental theorem of symmetric functions, P_j can be written as $P_j(\bar{x}_1, \ldots, \bar{x}_j; \bar{y}_1, \ldots, \bar{y}_j)$.

Thus, we can define on the multiplicative group

$$\bigwedge(A) = \{1 + a_1 t + a_2 t^2 + \cdots, a_i \in A\} \subseteq A[[t]]$$

a second operation by means of the formula

$$(1 + a_1 t + a_2 t^2 + \cdots) * (1 + b_1 t + b_2 t^2 + \cdots) = 1 + P_1(a,b) t + P_2(a,b) t^2 + \cdots$$

such that $(\bigwedge(A), \cdot, *)$ is a ring.

Moreover, with the previous notations we can define a map

$$\begin{aligned} E_A\colon \mathbb{W}(A) &\longrightarrow \bigwedge(A) \\ (a_1, a_2, \ldots) &\longrightarrow \prod_{i\geq 1}(1 - a_i t^i), \end{aligned}$$

which is an isomorphism of rings because $E_A(a+b) = E_A(a) \cdot E_A(b)$ and $E_A(a \cdot b) = E_A(a) * E_A(b)$.

Remark 2.2: Note that the map E_A is the generalization to arbitrary characteristic of the exponential map and that the sum in $\mathbb{W}(A)$ is analogous to the group operation induced by the Campbell–Hausdorff formula.

Let $\underline{\mathbb{W}}^\bullet\colon \mathcal{C}_{\text{sch.}/k} \longrightarrow \mathcal{C}_{\text{rings}}$ be the contravariant functor defined as

$$\underline{\mathbb{W}}^\bullet(S) = \mathbb{W}(H^0(S, \mathcal{O}_S)),$$

and let us denote by $\underline{\mathbb{W}}_+^\bullet$ the induced functor on the category of abelian groups.

It is clear that the above exponential gives an isomorphism of functors on groups $\underline{\mathbb{W}}_+^\bullet \simeq \Gamma_+^\bullet$.

Furthermore, for each ring B we can define the abelian group

$$\hat{\mathbb{W}}_+(B) = \left\{ \begin{array}{c} (b_1, b_2, \ldots) \in \mathbb{W}_+(B) \text{ with } b_i \text{ nilpotent} \\ \text{for all } i \text{ and } b_i = 0 \text{ for almost all } i \end{array} \right\}.$$

Analogously to the above case, we have that $\hat{\mathbb{W}}_+(H^0(S, \mathcal{O}_S)) \simeq \Gamma_-^\bullet(S)$, from which we obtain the existence of an isomorphism of functors on groups

$$\underline{\hat{\mathbb{W}}}_+^\bullet \simeq \Gamma_-^\bullet.$$

2.C. Duality between the group schemes Γ_+ and Γ_-. If Y is a group scheme over a field k, we define its functor of Cartier characters, $\chi(Y)^\bullet$, as

$$\chi(Y)^\bullet(S) = \operatorname{Hom}_{S-groups}(Y_S, (\mathbb{G}_m)_S),$$

S being a k-scheme and $\mathbb{G}_m$ being the multiplicative group scheme.

With the previous notations, for each k-scheme S we can establish a functorial map

$$\langle\ ,\ \rangle\colon \underline{\hat{\mathbb{W}}}_+^\bullet(S) \times \underline{\mathbb{W}}_+^\bullet(S) \longrightarrow \mathbb{G}_m^\bullet(S),$$

defined by

$$\langle a, b\rangle = E_{H^0(S,\mathcal{O}_S)}(a \cdot b; 1),$$

where $E_A(x;t) := E_A(x)[t]$, $a \in \hat{\mathbb{W}}_+(H^0(S,\mathcal{O}_S)$ and $b \in \mathbb{W}_+(H^0(S,\mathcal{O}_S)$.

For each ring B, one has that

$$\hat{\mathbb{W}}_+(B) = \mathrm{Hom}_{B-groups}((\mathbb{W}_+)_B, \mathbb{G}_{m,B})$$

([9], page 500), and hence $\underline{\hat{\mathbb{W}}}^\bullet_+(S) \simeq \chi(\Gamma_+)^\bullet(S)$. Thus, $\chi(\Gamma_+)^\bullet$ is representable by a formal group scheme, $\chi(\Gamma_+) \simeq \Gamma_-$, and, in arbitrary characteristic, the universal character determines a natural morphism of schemes

$$\begin{aligned} \chi\colon \Gamma_+ \times \Gamma_- &\longrightarrow \mathbb{G}_m \\ (f,g) &\longmapsto \chi_g(f). \end{aligned}$$

Moreover, since

$$\mathbb{W}_+(B) = \mathrm{Hom}_{B-groups}((\hat{\mathbb{W}}_+)_B, \mathbb{G}_{m,B}),$$

the functor $\chi(\Gamma_-)^\bullet$ is also representable; its representant is Γ_+ and the universal character is the above one. Accordingly, the groups Γ_+ and Γ_- are autodual.

Remark 2.3: If S is a connected k-scheme, $f = \prod_{i=1}^\infty (1 - a_i z^i) \in \Gamma^\bullet_+(S)$ and $g \in \prod_{j=1}^h (1 - b_{-j} z^{-j}) \in \Gamma^\bullet_-(S)$, where b_{-j} are nilpotent elements of $H^0(S,\mathcal{O}_S)$, one has that

$$\begin{aligned} \chi_g(f) &= E_{H^0(S,\mathcal{O}_S)}(a \cdot b; 1) = \prod_{i \geq 1} (1 - (a \cdot b)_i) \\ &= \prod_{i=1}^\infty \prod_{j=1}^h (1 - a_i^{j/(i,j)} b_{-j}^{i/(i,j)})^{(i,j)} \in H^0(S,\mathcal{O}_S)^* = \mathbb{G}^\bullet_m(S), \end{aligned}$$

where, finitely, many of the terms appearing in the products differ from 1.

Proposition 2.4: *If* $\mathrm{char}(k) = 0$, $f(t) \in \Gamma^\bullet_+(S)$ *and* $g(t) \in \Gamma^\bullet_-(S)$, *one has that*

$$\chi(f(t), g(t)) = \exp\Big[\mathrm{res}\Big(\log(f)\frac{dg}{g}\Big)\Big].$$

Proof: Given an element $h(t) \in A[[t]][t^{-1}]$ we set

$$\delta_m(h(t)) = \mathrm{res}\Big(t^m\Big(\frac{dh}{h}\Big)\Big) \quad (m \in \mathbb{Z}).$$

Then, if $f(t) = 1 + \sum_{i \geq 1} a_i t^i$ and $g(t) = 1 + \sum_{j=1}^N b_j t^{-j}$, such that

$$f(t) = \prod_{i \geq 1}(1 - \bar{a}_i t^i) = \exp\Big(\sum_{i \geq 1} \tilde{a}_i t^i\Big) \text{ and } g(t) = \prod_{j \geq 1}(1 - \bar{b}_j t^{-j}) = \exp\Big(\sum_{j \geq 1} \tilde{b}_j t^{-j}\Big),$$

we have the relations

$$\widetilde{a}_i = \frac{-\omega_i(\bar{a})}{i} = \frac{1}{i}\delta_i(f(t)) \quad \text{and} \quad \widetilde{b}_j = \frac{-\omega_j(\bar{b})}{j} = -\frac{1}{j}\delta_j(g(t)),$$

and hence

$$\begin{aligned}\chi(f(t), g(t)) &= E_{H^0(S,\mathcal{O}_S)}(\bar{a}\cdot\bar{b};1) = \prod_{i\geq 1}(1-(\bar{a}\cdot\bar{b})_i)\\ &= \exp\Big[\sum_{i\geq 1}\Big(\frac{-\omega_i(\bar{a}\cdot\bar{b})}{i}\Big)\Big] = \exp\Big[\sum_{i\geq 1}\Big(-\frac{1}{i}\omega_i(\bar{a})\cdot\omega_i(\bar{b})\Big)\Big]\\ &= \exp\Big[\sum_{i\geq 1}(\delta_{-i}(f(t))\cdot\delta_i(g(t))/i)\Big] = \exp\Big[\sum_{i\geq 1}(-i\cdot\widetilde{a}_i\cdot\widetilde{b}_i)\Big]\\ &= \exp\Big[\operatorname{res}\Big(\log(f)\frac{dg}{g}\Big)\Big]. \qquad \blacksquare\end{aligned}$$

Remark 2.5: Given the group scheme Γ, let

$$0 \to \mathbb{G}_m \to \widetilde{\Gamma} \to \Gamma \to 0$$

be the extension of groups defined by the automorphisms of the determinant bundle over an infinite Grassmannian ([3], page 23). If $e\colon \Gamma\times\Gamma \to \mathbb{G}_m$ is the commutator of this extension, one has that $e|_{\Gamma_+\times\Gamma_-}$ coincides with the above character. Moreover, a similar expression to the statement of the above proposition is obtained in ([13], page 22).

3. Generalized definition of the Contou-Carrère symbol

2.A. Definition of the Contou-Carrère symbol associated with a field k. It is known that the group schemes $\mathbb{Z}_*$ and $\mathbb{G}_m$ are autodual and their universal character is the group morphism

$$\begin{aligned}\mathbb{Z}_*^\bullet(S)\times\mathbb{G}_m^\bullet(S) &\longrightarrow \mathbb{G}_m^\bullet(S)\\ (\alpha,\lambda) &\longmapsto \lambda^\alpha\end{aligned}$$

for each connected k-scheme S.

Thus, from the duality between the group schemes Γ_+ and Γ_-, if we set $K_0 = \mathbb{G}_m\times\Gamma_+$, we have that $\hat{K_0} = \mathbb{Z}_*\times\Gamma_-$, $\hat{K_0}$ being the Cartier dual of the group scheme K_0.

Definition 3.1: We shall use the term Heisenberg group scheme associated with $\widetilde{\Gamma}$, denoting this by $\mathcal{H}(\widetilde{\Gamma})$, to refer to the scheme

$$\mathcal{H}(\widetilde{\Gamma}) = \mathbb{G}_m \times K_0 \times \hat{K}_0$$

together with the group law

$$(\alpha, x, l) \cdot (\alpha', x', l') = (\alpha \cdot \alpha' \cdot l(x'), x \cdot x', l \cdot l')$$

for S-valued points, S being a k-scheme.

If $e_{\mathcal{H}(\widetilde{\Gamma})}(x, y)$ is the commutator of $\mathcal{H}(\widetilde{\Gamma})$, and since $\widetilde{\Gamma} \simeq \chi(\widetilde{\Gamma})$, the map

$$\begin{aligned} \varphi\colon \widetilde{\Gamma} &\longrightarrow \chi(\widetilde{\Gamma}) \\ x &\longmapsto e_{\mathcal{H}(\widetilde{\Gamma})}(x, \quad) \end{aligned}$$

is an isomorphism of groups, because if $x = (a, b) \in K_0 \times \hat{K}_0$, then $\varphi(a, b) = (a^{-1}, b)$.

Thus, $\mathcal{H}(\widetilde{\Gamma})$ satisfies the characterization as a Heisenberg group of an extension by the multiplicative group ([11], page 2).

Moreover, since $e_{\mathcal{H}(\widetilde{\Gamma})}\colon \widetilde{\Gamma} \times \widetilde{\Gamma} \to \mathbb{G}_m$ is a 2-cocycle, it determines an element of the cohomology group $H^2_{\text{reg.}}(\widetilde{\Gamma}, \mathbb{G}_m)$. This group contains the classes of 2-cocycles that are morphisms of schemes. We shall now give a definition of the Contou-Carrère symbol as a morphism of schemes from this cohomology class.

LEMMA 3.2: *There exists a unique 2-coboundary $c\colon \mathbb{Z}_* \times \mathbb{Z}_* \to \mathbb{G}_m$ satisfying the conditions:*

- $c(\alpha, \beta + \gamma) = c(\alpha, \beta) \cdot c(\alpha, \gamma)$
- $c(\alpha, \alpha) = (-1)^\alpha$

for $\alpha, \beta, \gamma \in \mathbb{Z}_*^{\bullet}(S)$, with S a connected k-scheme.

Proof: Recall that a 2-cocycle $c\colon \mathbb{Z}_* \times \mathbb{Z}_* \to \mathbb{G}_m$ is a 2-coboundary when there exists a morphism of schemes $\psi\colon \mathbb{Z}_* \to \mathbb{G}_m$ such that

$$c(\alpha, \beta) = \psi(\alpha + \beta) \cdot \psi(\alpha)^{-1} \cdot \psi(\beta)^{-1}.$$

Then, since $\mathbb{Z}_* = \coprod_{\alpha \in \mathbb{Z}} \operatorname{Spec} k$, the morphism ψ is determined by a sequence $(\lambda_\alpha)_{\alpha \in \mathbb{Z}}$, with $\psi(\alpha) = \lambda_\alpha \in k^*$. It follows from the conditions of the lemma that

$$\lambda_\alpha = (-1)^{\alpha(\alpha-1)/2} \lambda_1^\alpha \quad \text{for all } \alpha \in \mathbb{Z}.$$

Hence, $c(\alpha, \beta) = (-1)^{\alpha \cdot \beta}$ is the unique 2-coboundary that satisfies the conditions of the Lemma. ∎

Definition 3.3: If S is a connected k-scheme and $f \in \widetilde{\Gamma}^\bullet(S) \simeq \mathbb{Z}_*^\bullet(S) \times \Gamma^\bullet(S)$, we shall call its component in $\mathbb{Z}_*^\bullet(S)$, which is an integer number, "the valuation of f" -$v(f)$-.

PROPOSITION 3.4: *There exists a unique 2-coboundary $\widetilde{c}\colon \widetilde{\Gamma} \times \widetilde{\Gamma} \to \mathbb{G}_m$ that satisfies the conditions:*

- $\widetilde{c}(f, g \cdot g') = \widetilde{c}(f,g) \cdot \widetilde{c}(f,g')$
- $\widetilde{c}(f,g) = 1$ *if* $v(f) = 0$
- $\widetilde{c}(f,-f) = (-1)^{v(f)}$

for $f, g, g' \in \widetilde{\Gamma}^\bullet(S)$, S being a connected k-scheme.

Proof: Since $\widetilde{c}$ is a 2-coboundary and $\widetilde{\Gamma}$ is commutative, one has that $\widetilde{c}(f,g) = \widetilde{c}(g,f)$. Hence $\widetilde{c}(f \cdot f', g) = \widetilde{c}(f,g) \cdot \widetilde{c}(f',g)$ and, if $v(g) = 0$, then $\widetilde{c}(f,g) = 1$.

Moreover, since $\widetilde{\Gamma}$ is a locally connected scheme, $\widetilde{c}$ is determined by considering S-valued points, with S a connected k-scheme. Let us now consider

$$f, g \in \widetilde{\Gamma}^\bullet(S) = \mathbb{Z}_*^\bullet(S) \times \Gamma^\bullet(S).$$

If we set $f = [\alpha, f_0]$ and $g = [\beta, g_0]$ with $v(f_0) = v(g_0) = 0$, one has that

$$\widetilde{c}(f,g) = \widetilde{c}([\alpha, f_0], [\beta, g_0]) = \widetilde{c}([\alpha, 1], [\beta, 1])$$

and thus there exists a 2-coboundary $c_1\colon \mathbb{Z}_* \times \mathbb{Z}_* \to \mathbb{G}_m$ such that

$$\widetilde{c}([\alpha, f_0], [\beta, g_0]) = c_1(\alpha, \beta).$$

Finally, bearing in mind that

$$\widetilde{c}([\alpha, f_0], [\alpha, -f_0]) = (-1)^\alpha = c_1(\alpha, \alpha),$$

it follows from the above Lemma that c_1 is unique and its value is

$$c_1(\alpha, \beta) = (-1)^{\alpha \cdot \beta} = \widetilde{c}(f,g). \qquad \blacksquare$$

THEOREM 3.5: *There exists a unique element $(\ , \ )_k$ in the cohomology class $[e_{\mathcal{H}(\widetilde{\Gamma})}] \in H^2_{reg}(\widetilde{\Gamma}, \mathbb{G}_m)$ satisfying the conditions:*

- $(f, g \cdot g')_k = (f,g)_k \cdot (f,g')_k$
- $(f,g)_k = e_{\mathcal{H}(\widetilde{\Gamma})}(f,g)$ *if* $v(f) = 0$
- $(f,-f)_k = 1$

for $f, g, g' \in \widetilde{\Gamma}^\bullet(S)$, with S a connected k-scheme. This element is the Contou-Carrère symbol associated with the field k.

Proof: The equality $e_{\mathcal{H}(\widetilde{\Gamma})}(f, g \cdot g') = e_{\mathcal{H}(\widetilde{\Gamma})}(f,g) \cdot e_{\mathcal{H}(\widetilde{\Gamma})}(f,g')$ follows from the properties of the commutator.

Moreover, if $f = (\alpha, \lambda, f_1, f_2)$ with $\alpha \in \mathbb{Z}$, $\lambda \in \mathbb{G}_m^\bullet(S)$, $f_1 \in \Gamma_+^\bullet(S)$ and $f_2 \in \Gamma_-^\bullet(S)$, hence $-f = (\alpha, -\lambda, f_1, f_2)$ and then

$$e_{\mathcal{H}(\tilde{\Gamma})}(f, -f) = \frac{\lambda^\alpha \chi(f_1, f_2)}{(-\lambda)^\alpha \chi(f_1, f_2)} = (-1)^\alpha.$$

Thus, if $\widetilde{c}$ is the 2-coboundary computed in Proposition 3.4, the 2-cocycle

$$(f, g)_k = \widetilde{c}(f, g) \cdot e_{\mathcal{H}(\tilde{\Gamma})}(f, g)$$

is the only one that satisfies the conditions of the Theorem. ∎

Remark 3.6: The generalization of the Contou-Carrère symbol that we have defined is determined by a cohomology class

$$[e_{\mathcal{H}(\tilde{\Gamma})}] \in H^2_{reg}(\widetilde{\Gamma}, \mathbb{G}_m),$$

and by three conditions which imply the uniqueness of a representant. These conditions are not strange in the theory of symbols because the first and the second appear in Serre's definition of the multiplicative local symbol [14] and the third is one of the properties verified by Steinberg symbols [10].

This symbol determines a central extension of $\widetilde{\Gamma}$ by $\mathbb{G}_m$ that coincides with the extension induced by the commutator $e_{\mathcal{H}(\tilde{\Gamma})}$. Let us denote this extension by $\widetilde{\Gamma}_{e_{\mathcal{H}(\tilde{\Gamma})}}$.

Recently, A. Beilinson, S. Bloch and H. Esnault [6] have defined the Contou-Carrère symbol as the commutator pairing in a Heisenberg super extension. Indeed, keeping the notations of [6], putting $F = k((t))$ and setting

$$F^x = \mathbb{Z} \times \mathbb{G}_m \times \mathbb{W} \times \hat{\mathbb{W}},$$

we can assign a Heisenberg super extension of F^x to any 1-dimensional vector space L over the local field F. Namely, considering L as a Tate vector space over the base field k, it yields the Tate super extension of the group $\mathrm{Gl}(L)$ of continuous k-automorphisms of L, and the BBE Heisenberg super extension is its pull-back to the subgroup of F-homotheties $F^x \subset \mathrm{Gl}(L)$. Since the construction is natural, the isomorphism class of the extension does not depend on the choice of L. Thus, the commutator pairing is equal to the Contou-Carrère symbol for any L. Hence, if we take for L a square root of the line of 1-forms, then the corresponding Heisenberg extension $F^{x\flat}$ naturally acquires a symmetric structure.

Thus, bearing in mind the characterization of symmetric Heisenberg extensions offered in [5], we have that $\widetilde{\Gamma}_{e_{\mathcal{H}(\tilde{\Gamma})}}$ coincides with the Baer product of two copies of F^{xb}.

COROLLARY 3.7: *If S is a connected k-scheme and $u, v \in \widetilde{\Gamma}^{\bullet}(S)$ with*

$$u = \lambda z^n \prod_{i=1}^{l}(1 - a_{-i}z^{-i}) \prod_{i=1}^{\infty}(1 - a_i z^i) \text{ and } v = \mu z^m \prod_{j=1}^{h}(1 - b_{-j}z^{-j}) \prod_{j=1}^{\infty}(1 - b_j z^j),$$

a_{-i}, b_{-j} being nilpotent elements of $H^0(S, \mathcal{O}_S)$, one has that

$$(u, v)_k = (-1)^{n \cdot m} \frac{\lambda^m \cdot \prod_{i=1}^{\infty} \prod_{j=1}^{h} (1 - a_i^{j/(i,j)} b_{-j}^{i/(i,j)})^{(i,j)}}{\mu^n \cdot \prod_{j=1}^{\infty} \prod_{i=1}^{l} (1 - b_j^{i/(i,j)} a_{-i}^{j/(i,j)})^{(i,j)}},$$

where, finitely, many of the terms appearing in the products differ from 1.

Proof: Since $\tilde{c}(u, v) = (-1)^{n \cdot m}$, the claim is directly deduced from the definition of the Heisenberg group and the value of the respective dual morphisms. ∎

COROLLARY 3.8: *Let us assume that* $\mathrm{char}(k) = 0$ *together with the hypothesis of Corollary 3.7. Hence*

$$(u, v)_k = (-1)^{n \cdot m} \frac{\lambda^m \cdot \exp(\sum_{i>0}(\delta_i(u) \cdot \delta_{-i}(v)/i))}{\mu^n \cdot \exp(\sum_{i>0}(\delta_{-i}(u) \cdot \delta_i(v)/i))},$$

where $\delta_s(f) = \mathrm{res}(z^s \cdot \frac{df}{f})$. This formula is equal to the expression obtained by C. Contou-Carrère in [7].

Proof: If we set $u_1 = \prod_{j=1}^{h}(1 - a_{-i}z^{-i})$ and $u_2 = \prod_{i=1}^{\infty}(1 - a_i z^i)$, for $i > 0$, we have that $\delta_i(u) = \delta_i(u_1)$ and $\delta_{-i}(u) = \delta_{-i}(u_2)$. And the same applies to v. Then, writing $v_1 = \prod_{j=1}^{h}(1 - b_{-j}z^{-j})$ and $v_2 = \prod_{j=1}^{\infty}(1 - b_j z^j)$, from Proposition 2.4 we have that

$$e_{\mathcal{H}(\tilde{\Gamma})}(u, v) = \frac{\chi_\lambda(m) \cdot \chi_{u_1}(v_2)}{\chi_\mu(n) \cdot \chi_{v_1}(u_2)} = \frac{\lambda^m \cdot \exp(\sum_{i>0}(\delta_i(u_1) \cdot \delta_{-i}(v_2)/i))}{\mu^n \cdot \exp(\sum_{i>0}(\delta_{-i}(u_2) \cdot \delta_i(v_1)/i))}.$$

Hence,

$$(u, v)_k = (-1)^{n \cdot m} \frac{\lambda^m \cdot \exp(\sum_{i>0}(\delta_i(u) \cdot \delta_{-i}(v)/i))}{\mu^n \cdot \exp(\sum_{i>0}(\delta_{-i}(u) \cdot \delta_i(v)/i))}.$$ ∎

Let $p \in C$ be a non-singular and rational point on a connected curve over $\mathrm{Spec}\, k$. Since $(\hat{\mathcal{O}}_p)_0^* \simeq \widetilde{\Gamma}^{\bullet}(\mathrm{Spec}\, k)$, one has that $\Sigma_C^* \hookrightarrow \widetilde{\Gamma}^{\bullet}(\mathrm{Spec}\, k)$.

COROLLARY 3.9: *In an arbitrary characteristic, considering rational points, if* $f, g \in \Sigma_C^*$ *one has that*

$$(f,g)_k = (f,g)_p = (-1)^{n \cdot m} \frac{f^n}{g^m}(p) \quad \text{with } n = v_p(g) \text{ and } m = v_p(f),$$

which is the expression of the multiplicative local symbol defined by J. P. Serre.

Proof: The claim results directly from $\Gamma_{-}^{\bullet}(\operatorname{Spec} k) = \{1\}$. ∎

Remark 3.10: We should note that the above definition of the multiplicative local symbol, as a map induced between the Spec k-valued points of a morphism of schemes, is "local", while Serre's definition of this symbol contains, among the conditions that imply its uniqueness, the reciprocity law $\prod_{p \in C}(f,g)_p = 1$, which is not a local condition.

Let us now consider a connected curve $\bar{C}$ over Spec k, and a non-singular point $\bar{p} \in \bar{C}$ such that $k \hookrightarrow k(\bar{p})$ is a finite and separable extension, $k(\bar{p})$ being the residue class field of $\bar{p}$. From the Theorem of Cohen we have that $(\hat{\mathcal{O}}_{\bar{p}})_0^* \simeq \widetilde{\Gamma}^{\bullet}(\operatorname{Spec} k(\bar{p}))$, and hence $\Sigma_C^* \hookrightarrow \widetilde{\Gamma}^{\bullet}(\operatorname{Spec} k(\bar{p}))$.

COROLLARY 3.11: *In an arbitrary characteristic, considering* Spec $k(\bar{p})$*-valued points, if* $\bar{f}, \bar{g} \in \Sigma_C^*$ *one has that*

$$(\bar{f},\bar{g})_k = (\bar{f},\bar{g})_{\bar{p}} = (-1)^{v_{\bar{p}}(\bar{f}) \cdot v_{\bar{p}}(\bar{g})} \frac{\bar{f}^{v_{\bar{p}}(\bar{g})}}{\bar{g}^{v_{\bar{p}}(\bar{f})}}(p) \in k(\bar{p})^*,$$

which is the expression of the tame symbol defined by J. Milnor and associated with the discrete valuation $v_{\bar{p}}$ *on the field* Σ_C^*.

2.B. GENERALIZATION OF THE CONTOU-CARRÈRE SYMBOL ASSOCIATED WITH A SEPARABLE EXTENSION $k \hookrightarrow k(s)$. We shall now give a generalization of the Contou-Carrère symbol associated with a separable extension $k \hookrightarrow k(s)$. To do so, we shall first construct a formal group scheme, Γ_K, from a deformation of the functor of points $\Gamma^{\bullet}$ induced by a finite extension $k \hookrightarrow K$.

PROPOSITION 3.12: *Let* $k \hookrightarrow K$ *be a finite extension. One has that the functor*

$$F_1(S) = \mathbb{G}_m^{\bullet}(S \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} K)$$

is representable in the category of k*-schemes and its representant, which is a group scheme, will be denoted by* $(\mathbb{G}_m)_K$.

Proof: Let us consider $K^{\vee} = \operatorname{Hom}_{k-mod.}(K,k)$, which is a finite k-module.

If $V(K) = \operatorname{Spec} S^{\cdot}(K^{\vee})$ is the vector bundle associated with it, one has that

$$\begin{aligned} V(K)^{\bullet}(S) &= \operatorname{Hom}_{k-alg.}(S^{\cdot}(K^{\vee}), H^0(S, \mathcal{O}_S)) \\ &= \operatorname{Hom}_{k-mod}(K^{\vee}, H^0(S, \mathcal{O}_S) \\ &= \operatorname{Hom}_{k-mod}(k, H^0(S, \mathcal{O}_S) \underset{k}{\otimes} K) \\ &= \operatorname{Hom}_{k-alg.}(k[x], H^0(S, \mathcal{O}_S) \underset{k}{\otimes} K) \end{aligned}$$

for each k-scheme S.

Then,

$$V(K)^{\bullet}(S) = \mathbb{G}_a^{\bullet}(S \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} K) = H^0(S, \mathcal{O}_S) \underset{k}{\otimes} K.$$

Hence, if $P(K)$ is the principal bundle obtained as the complement of the 0-section in $V(K)$, we have that

$$P(K)^{\bullet}(S) = (H^0(S, \mathcal{O}_S) \underset{k}{\otimes} K)^* = F_1(S),$$

from where we deduce that F_1 is representable. ∎

PROPOSITION 3.13: *Let $k \hookrightarrow K$ be a finite extension. One has that the functor*

$$F_2(S) = \Gamma_+^{\bullet}(S \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} K)$$

is representable in the category of k-schemes. Its representant will be denoted by $(\Gamma_+)_K$ and, by construction, is a group scheme.

Proof: Let us consider the ring

$$B_n = S^{\cdot}(K^{\vee}) \underset{k}{\otimes} \overset{n)}{\cdots} \underset{k}{\otimes} S^{\cdot}(K^{\vee}) = S^{\cdot}((k \oplus \overset{n)}{\cdots} \oplus k) \underset{k}{\otimes} K^{\vee}).$$

Arguing similarly to the above proposition, one has that:

$$\operatorname{Hom}_{k-alg}(B_n, H^0(S, \mathcal{O}_S)) = \operatorname{Hom}_{k-alg}(k[x_1, \ldots, x_n], H^0(S, \mathcal{O}_S) \underset{k}{\otimes} K),$$

and hence

$$(\operatorname{Spec} B_n)^{\bullet}(S) = (\mathbb{A}^n)^{\bullet}(S \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} K).$$

Thus, we have that

$$(\Gamma_+)_K = \varinjlim_n \operatorname{Spec} B_n = \operatorname{Spec}(\varprojlim_n B_n),$$

together with the corresponding group law. F_2 is therefore representable. ∎

PROPOSITION 3.14: *Let $k \hookrightarrow K$ be a finite extension. One has that the functor*

$$F_3(S) = \Gamma^\bullet_-(S \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} K)$$

is representable in the category of formal k-schemes. We shall denote by $(\Gamma_-)_K$ its representant, which is a formal group scheme.

Proof: If $(\mathbb{A}^n)_K = \operatorname{Spec} B_n$ is the scheme studied in the previous Proposition, we have a natural morphism of schemes $\mathbb{A}^n \hookrightarrow (\mathbb{A}^n)_K$ defined between the S-valued points as the inclusion

$$\bigoplus_n H^0(S, \mathcal{O}_S) \hookrightarrow \bigoplus_n H^0(S, \mathcal{O}_S) \underset{k}{\otimes} K.$$

We thus have a natural ring morphism $B_n \longrightarrow k[x_1, \dots, x_n]$ and there exists a maximal ideal $\widetilde{\mathfrak{m}}_0 \subseteq B_n$ such that $\widetilde{\mathfrak{m}}_0 \cdot k[x_1, \dots, x_n] = (x_1, \dots, x_n)$.

Hence,

$$k[x_1, \dots, x_n]/(x_1, \dots, x_n)^n \simeq k[x_1, \dots, x_n] \underset{B_n}{\otimes} B_n/\widetilde{\mathfrak{m}}_0^n$$

and we deduce that

$$\begin{aligned} &\operatorname{Hom}_{k-alg}(k[x_1, \dots, x_n]/(x_1, \dots, x_n)^n, H^0(S, \mathcal{O}_S) \underset{k}{\otimes} K) \\ &= \operatorname{Hom}_{k-alg}(k[x_1, \dots, x_n] \underset{B_n}{\otimes} B_n/\widetilde{\mathfrak{m}}_0^n, H^0(S, \mathcal{O}_S) \underset{k}{\otimes} K) \\ &= \operatorname{Hom}_{k-alg}(B_n, H^0(S, \mathcal{O}_S)) \underset{\operatorname{Hom}_{k-alg}(B_n, H^0(S, \mathcal{O}_S) \underset{k}{\otimes} K)}{\times} \operatorname{Hom}_{k-alg}(B_n/\widetilde{\mathfrak{m}}_0^n, H^0(S, \mathcal{O}_S) \underset{k}{\otimes} K) \\ &= \operatorname{Hom}_{k-alg}(B_n/\widetilde{\mathfrak{m}}_0^n, H^0(S, \mathcal{O}_S)). \end{aligned}$$

Thus,

$$\begin{aligned} &\operatorname{Spec}(B_n/\widetilde{\mathfrak{m}}_0^n)^\bullet(S) \\ &\quad = \operatorname{Hom}_{k-alg}(k[x_1, \dots, x_n]/(x_1, \dots, x_n)^n, H^0(S, \mathcal{O}_S) \underset{k}{\otimes} K), \end{aligned}$$

and introducing the corresponding structure of a Witt group scheme in $(\mathbb{A}^n)_K$, we have that

$$\operatorname{Spec}(B_n/\widetilde{\mathfrak{m}}_0^n)^\bullet(S) = \left\{ \begin{array}{l} a_n z^{-n} + \cdots + a_1 z^{-1} + 1 \text{ where} \\ a_i \in H^0(S, \mathcal{O}_S) \underset{k}{\otimes} K \text{ and } (a_1, \dots, a_n)^n = 0 \end{array} \right\},$$

from where we deduce that the functor F_3 is representable by the formal group scheme

$$(\Gamma_-)_K = \varprojlim_n \operatorname{Spec}(B_n/\widetilde{\mathfrak{m}}_0^n). \qquad \blacksquare$$

Let us now consider a finite and separable extension $k \hookrightarrow k(s)$. We set $\widetilde{\Gamma_{k(s)}} = \mathbb{Z}_* \times \Gamma_{k(s)}$, where $\Gamma_{k(s)} = (\Gamma_+)_{k(s)} \times (\mathbb{G}_m)_{k(s)} \times (\Gamma_-)_{k(s)}$, which is a locally connected scheme. Moreover, for each connected k-scheme S, one has that

$$\widetilde{\Gamma_{k(s)}}^{\bullet}(S) = (H^0(S, \mathcal{O}_S) \underset{k}{\otimes} k(s)[[z]][z^{-1}])^*,$$

and hence $\widetilde{\Gamma_{k(s)}}^{\bullet}(\operatorname{Spec} k) = (k(s)[[z]][z^{-1}])^* = (k(s)[[z]])^*_{(0)}$.

If we denote $G_{k(s)} = (\mathbb{G}_m)_{k(s)} \times (\Gamma_+)_{k(s)}$ and $\overline{G_{k(s)}} = \mathbb{Z}_* \times (\Gamma_-)_{k(s)}$, from the natural inclusion of functors on groups

$$\mathbb{Z}_*^{\bullet}(S) \hookrightarrow (\mathbb{Z}_*)^{\bullet}_{k(s)}(S) = \mathbb{Z}_*^{\bullet}(S \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} k(s))$$

we have defined a morphism of schemes

$$\chi_{k(s)}\colon G_{k(s)} \times \overline{G_{k(s)}} \longrightarrow (\mathbb{G}_m)_{k(s)}$$

induced by the duality between the group schemes $\mathbb{G}_m \times \Gamma_+$ and $\mathbb{Z}_* \times \Gamma_-$.

Let us now consider the norm morphism $N_{k(s)/k}\colon (\mathbb{G}_m)_{k(s)} \to \mathbb{G}_m$, which is defined between the S-valued points as

$$\begin{aligned} N_{k(s)/k}\colon (\mathbb{G}_m)^{\bullet}_{k(s)}(S) &\longrightarrow \mathbb{G}_m^{\bullet}(S) \\ f &\longmapsto \det h_f, \end{aligned}$$

where

$$h_f\colon H^0(S, \mathcal{O}_S) \underset{k}{\otimes} k(s) \to H^0(S, \mathcal{O}_S) \underset{k}{\otimes} k(s)$$

is the homothety induced by

$$f \in (H^0(S, \mathcal{O}_S) \underset{k}{\otimes} k(s))^*$$

and $\det h_f$ is computed from a basis of

$$H^0(S, \mathcal{O}_S) \underset{k}{\otimes} k(s)$$

as a finite $H^0(S, \mathcal{O}_S)$-module.

Hence, we have a morphism of schemes $\bar{\chi}\colon G_{k(s)} \times \overline{G_{k(s)}} \longrightarrow \mathbb{G}_m$ defined by

$$\bar{\chi}(f, g) = N_{k(s)/k}(\chi_{k(s)}(f, g)).$$

Our aim is now to define a group scheme associated with $\widetilde{\Gamma_{k(s)}}$ that will generalize the notion of the Heisenberg group scheme studied in the previous subsection. To do this, we shall denote by $\bar{\mathcal{H}}(\widetilde{\Gamma_{k(s)}})$ the group scheme $\mathbb{G}_m \times G_{k(s)} \times \overline{G_{k(s)}}$ defined from the operation

$$(\alpha, f, g) \cdot (\alpha', f', g') = (\alpha \cdot \alpha' \cdot \bar{\chi}(f', g), f \cdot f', g \cdot g')$$

for S-valued points, S being a k-scheme.

Moreover, if $e_{\bar{\mathcal{H}}(\widetilde{\Gamma_{k(s)}})}\colon \widetilde{\Gamma_{k(s)}} \times \widetilde{\Gamma_{k(s)}} \longrightarrow \mathbb{G}_m$ is the commutator of the extension of groups induced by $\bar{\mathcal{H}}(\widetilde{\Gamma_{k(s)}})$, one has that

$$e_{\bar{\mathcal{H}}(\widetilde{\Gamma_{k(s)}})}[(f,g),(f',g')] = N_{k(s)/k}\Big(\frac{\chi_{k(s)}(f',g)}{\chi_{k(s)}(f,g')}\Big).$$

Since $e_{\bar{\mathcal{H}}(\widetilde{\Gamma_{k(s)}})}$ determines an element of the group $H^2_{reg}(\widetilde{\Gamma_{k(s)}}, \mathbb{G}_m)$, we shall define a new generalization of the tame symbol similarly to the previous one.

Let us denote $\deg(k(s)) = \dim_k k(s)$.

LEMMA 3.15: *There exists a unique 2-coboundary $c\colon \mathbb{Z}_* \times \mathbb{Z}_* \to \mathbb{G}_m$ satisfying the conditions:*

- $c(\alpha, \beta+\gamma) = c(\alpha,\beta)\cdot c(\alpha,\gamma)$
- $c(\alpha,\alpha) = (-1)^{\alpha\cdot\deg(k(s))}$

for $\alpha,\beta,\gamma \in \mathbb{Z}_*^\bullet(S)$, with S a connected k-scheme.

Proof: Analogously to the proof of Lemma 3.2, one has that

$$c(\alpha,\beta) = (-1)^{\alpha\cdot\beta\cdot\deg(k(s))}$$

is the unique 2-coboundary that satisfies the conditions of the Lemma. ∎

PROPOSITION 3.16: *There exists a unique 2-coboundary*

$$\widetilde{c}\colon \widetilde{\Gamma_{k(s)}} \times \widetilde{\Gamma_{k(s)}} \to \mathbb{G}_m$$

that satisfies the conditions:

- $\widetilde{c}(f, g\cdot g') = \widetilde{c}(f,g)\cdot\widetilde{c}(f,g')$
- $\widetilde{c}(f,g) = 1$ *if* $v(f) = 0$
- $\widetilde{c}(f,-f) = (-1)^{v(f)\cdot\deg(k(s))}$

for $f,g,g' \in \widetilde{\Gamma_{k(s)}}^\bullet(S)$, S being a connected k-scheme.

Proof: The proof is similar to the proof of Proposition 3.4. ∎

THEOREM 3.17: *There exists a unique element $(\,,\,)_{k(s)}$ in the cohomology class $[e_{\bar{\mathcal{H}}(\widetilde{\Gamma_{k(s)}})}] \in H^2_{reg}(\widetilde{\Gamma_{k(s)}}, \mathbb{G}_m)$ satisfying the conditions*

- $(f, g\cdot g')_{k(s)} = (f,g)_{k(s)}\cdot(f,g')_{k(s)}$
- $(f,g)_{k(s)} = e_{\bar{\mathcal{H}}(\widetilde{\Gamma_{k(s)}})}(f,g)$ *if* $v(f) = 0$
- $(f,-f)_{k(s)} = 1$

for $f, g, g' \in \widetilde{\Gamma_{k(s)}}^{\bullet}(S)$, with S a connected k-scheme. This element is a generalization of the Contou-Carrère symbol associated with the separable extension $k \hookrightarrow k(s)$.

Proof: Arguing identically to the proof of Theorem 3.5, one sees that

$$(f,g)_{k(s)} = \widetilde{c}(f,g) \cdot e_{\tilde{\mathcal{H}}(\widetilde{\Gamma_{k(s)}})}(f,g)$$

is the only element that satisfies the conditions of the Theorem. ∎

COROLLARY 3.18: *If S is a connected k-scheme and $u, v \in \widetilde{\Gamma_{k(s)}}^{\bullet}(S)$ with*

$$u = \lambda z^n \prod_{i=1}^{l}(1 - a_{-i}z^{-i}) \prod_{i=1}^{\infty}(1 - a_i z^i) \text{ and } v = \mu z^m \prod_{j=1}^{h}(1 - b_{-j}z^{-j}) \prod_{j=1}^{\infty}(1 - b_j z^j),$$

a_{-i}, b_{-j} being nilpotent elements of $H^0(S, \mathcal{O}_S) \otimes_k k(s)$, one has that

$$(u,v)_{k(s)} = (-1)^{n\cdot m\cdot deg(k(s))} N_{k(s)/k}\Big(\frac{\lambda^m \prod_{i=1}^{\infty} \prod_{j=1}^{h}(1 - a_i^{j/(i,j)} b_{-j}^{i/(i,j)})^{(i,j)}}{\mu^n \prod_{j=1}^{\infty} \prod_{i=1}^{l}(1 - b_j^{i/(i,j)} a_{-i}^{j/(i,j)})^{(i,j)}}\Big),$$

where, finitely, many of the terms appearing in the products differ from 1.

Proof: The result is immediately deduced from the value of the generalized symbol showed in Corollary 3.7. ∎

COROLLARY 3.19: *If we add the condition* $\operatorname{char}(k) = 0$ *to the hypothesis of the previous corollary, we have that*

$$(u,v)_{k(s)} = (-1)^{n\cdot m\cdot \deg(k(s))} N_{k(s)/k}\Big(\frac{\lambda^m \cdot \exp(\sum_{i>0}(\delta_i(u)\cdot \delta_{-i}(v)/i))}{\mu^n \cdot \exp(\sum_{i>0}(\delta_{-i}(u)\cdot \delta_i(v)/i))}\Big),$$

where $\delta_s(f) = \operatorname{res}(z^s \cdot \frac{df}{f})$.

Proof: The statement is a direct consequence of Corollary 3.8 ∎

Remark 3.20: Let C be an irreducible and non-singular curve over a perfect field k and let $p \in C$ be a closed point on it. If $k(p)$ is the residue class field of p, one has that $k \hookrightarrow k(p)$ is a separable extension and, from the Theorem of Cohen, one deduces that $\hat{\mathcal{O}}_p \simeq k(p)[[z]]$. Then, $\Sigma_C^* \hookrightarrow \widetilde{\Gamma_{k(p)}}^{\bullet}(\operatorname{Spec} k)$. Accordingly, if $f, g \in \Sigma_C^*$, one has that

$$(f,g)_{k(p)} = (-1)^{v_p(f)\cdot v_p(g)\cdot deg(k(p))} N_{k(p)/k}\Big(\frac{f^{v_p(g)}}{g^{v_p(f)}}(p)\Big),$$

which coincides with the expression obtained in [12].

4. Reciprocity law

The goal of this final section is to prove a reciprocity law for the above generalized Contou-Carrère symbol in terms of morphisms of schemes.

Let us again consider an irreducible, complete and non-singular curve C over a perfect field k, whose function field is Σ_C.

If p is a closed point of C, retaining the above notations, from the Theorem of Cohen one has that $\hat{\mathcal{O}}_p \simeq k(p)[[z]]$ and $\Sigma_C^* \hookrightarrow \widetilde{\Gamma_{k(p)}}^\bullet(\operatorname{Spec} k)$, where $\widetilde{\Gamma_{k(p)}}$ is the k-formal scheme associated with separable extension $k \hookrightarrow k(p)$ defined in subsection 3.B. Analogously, if A is a k-algebra, it is clear that $(\Sigma_C \otimes_k A)^* \hookrightarrow \widetilde{\Gamma_{k(p)}}^\bullet(\operatorname{Spec} A)$ for each $p \in C$.

Let us now consider the k-formal scheme

$$\prod_{p\in C}{}' \widetilde{\Gamma_{k(p)}} = \Big(\prod_p G_{k(p)}\Big) \times \Big(\bigoplus_p \overline{G}_{k(p)}\Big),$$

where $G_{k(p)}$ and $\overline{G}_{k(p)}$ are the k-schemes $G_{k(p)} = (\mathbb{G}_m)_{k(p)} \times (\Gamma_+)_{k(p)}$ and $\overline{G}_{k(p)} = \mathbb{Z}_* \times (\Gamma_-)_{k(p)}$.

For each connected k-scheme S, we have defined a morphism of groups

$$\begin{aligned}\Big(\prod_{p\in C}{}' \widetilde{\Gamma_{k(p)}}\Big)^\bullet(S) \times \Big(\prod_{p\in C}{}' \widetilde{\Gamma_{k(p)}}\Big)^\bullet(S) &\xrightarrow{\phi_S} (\mathbb{G}_m)^\bullet(S)\\ (\{f_p\},\{g_p\}) &\longmapsto \prod_p (f_p, g_p)_{k(p)},\end{aligned}$$

where $(f_p, g_p)_{k(p)} = 1$ for almost all closed points $p \in C$.

THEOREM 4.1 (Reciprocity Law for the Generalized Contou-Carrère Symbol): *For each artinian local finite k-algebra A, the natural morphism of groups*

$$(\Sigma_C \otimes_k A)^* \times (\Sigma_C \otimes_k A)^* \hookrightarrow \Big(\prod_{p\in C}{}' \widetilde{\Gamma_{k(p)}}\Big)^\bullet(\operatorname{Spec} A) \times \Big(\prod_{p\in C}{}' \widetilde{\Gamma_{k(p)}}\Big)^\bullet(\operatorname{Spec} A)$$

takes values on $\operatorname{Ker} \phi_{\operatorname{Spec} A}$.

Proof: Retain the notations of [1] and let us consider an artinian local finite k-algebra A. If p is a closed point of C and we set

$$B_p = \hat{\mathcal{O}}_p \otimes_k A \simeq (A \otimes_k k(p))[[z]],$$

and $K_p = (\hat{\mathcal{O}}_p)_0 \otimes_k A \simeq (A \otimes_k k(p))((z))$ (with $(\hat{\mathcal{O}}_p)_0$ the field of fractions of $\hat{\mathcal{O}}_p$), it follows from [1] (Section 3) that we have a central extension of groups

$$1 \to A^* \longrightarrow \widetilde{G}_{B_p}^{K_p} \longrightarrow G_{B_p}^{K_p} \to 1,$$

which induces by restriction another central extension

$$1 \to A^* \longrightarrow \widetilde{(\Sigma_C \otimes_k A)^*} \longrightarrow (\Sigma_C \otimes_k A)^* \to 1,$$

whose commutator is denoted by $\{f,g\}_{B_p}^{K_p}$ for all $f,g \in (\Sigma_C \otimes_k A)^*$.

Moreover, when $a,b \in A \otimes_k k(p)$ and b is a nilpotent element, from the computations conducted in [1] one has that

$$\begin{aligned}\{1-az^i, 1-bz^{-j}\}_{B_p}^{K_p} &= \det_A(1-az^i \mid (A\otimes_k k(p))[[z]]/(z^j-b)) \\ &= \det_A(\det_{A\otimes_k k(p)}\big(1-az^i \mid (A\otimes_k k(p))[[z]]/(z^j-b)\big) \mid A\otimes_k k(p)) \\ &= N_{k(p)/k}((1-a^{j/(i,j)}b^{i/(i,j)})^{(i,j)}),\end{aligned}$$

and thus if we consider $f,g \in (\Sigma_C \otimes_k A)^*$ such that

$$f = \lambda z^{v_p(f)} \prod_{i=1}^{l}(1-a_{-i}z^{-i})\prod_{i=1}^{\infty}(1-a_i z^i)$$

and

$$g = \mu z^{v_p(g)} \prod_{j=1}^{h}(1-b_{-j}z^{-j})\prod_{j=1}^{\infty}(1-b_j z^j),$$

a_{-i}, b_{-j} being nilpotent elements of $A \otimes_k k(p)$, we deduce that

$$\{f,g\}_{B_p}^{K_p} = N_{k(p)/k}\Big(\frac{\lambda^{v_p(g)}\prod_{i=1}^{\infty}\prod_{j=1}^{h}(1-a_i^{j/(i,j)}b_{-j}^{i/(i,j)})^{(i,j)}}{\mu^{v_p(f)}\prod_{j=1}^{\infty}\prod_{i=1}^{l}(1-b_j^{i/(i,j)}a_{-i}^{j/(i,j)})^{(i,j)}}\Big).$$

Hence $(f,g)_{k(p)} = (-1)^{v_p(f)\cdot v_p(g)\cdot \deg(k(p))}\{f,g\}_{B_p}^{K_p}$.

Finally, bearing in mind that

$$\operatorname{ind}_{B_p}^{K_p} f = v_p(f)\cdot \deg(k(p)) \quad \text{for all } f \in (\Sigma_C \otimes_k A)^*,$$

with similar arguments to Tate's proof of the residue theorem [15], one sees that

$$\prod_{p\in C}\{f,g\}_{B_p}^{K_p} = (-1)^{\sum_{p\in C} v_p(f)\cdot v_p(g)\cdot \deg(k(p))},$$

and we conclude that $\prod_{p\in C}(f,g)_{k(p)} = 1$. ∎

Remark 4.2: The above proof of the reciprocity law is based on the results of [1], where A must be an artinian local finite k-algebra. From the statements proved in [6] (Section 3.4), one would think that this formula remains true for a commutative k-algebra A (with $\Sigma_C \otimes_k A$ replaced by the ring of functions on the complement to a relative divisor). We are planning to study this generalization in depth in a future work, with the aim of understanding the results of [6] in terms of commensurable A-submodules.

Remark 4.3 (Tame and Hilbert Reciprocity Laws): If C satisfies the hypothesis of the theorem, by taking rational points in the above expression we can recover the reciprocity law of the tame symbol of an algebraic curve [12]:

$$\prod_{p\in C}(-1)^{v_p(f)\cdot v_p(g)\cdot \deg(k(p))}N_{k(p)/k}\Big(\frac{f^{v_p(g)}}{g^{v_p(f)}}(p)\Big)=1.$$

Similarly, if C is an irreducible, complete and non-singular curve over a finite perfect field that contains the mth roots of unity and $\sharp k = q$, the reciprocity law of the Hilbert norm residue symbol

$$\prod_{p\in C}N_{k(p)/k}\Big((-1)^{v_p(f)\cdot v_p(g)}\cdot\frac{f^{v_p(g)}}{g^{v_p(f)}}(p)\Big)^{(q-1)/m}=1$$

can also be deduced from the statement of the theorem.

Remark 4.4 (Generalized Residue Theorem): If C is again an irreducible, complete and non-singular curve over a perfect field, $p\in C$ is a closed point, and $f,g\in\Sigma_C^*\hookrightarrow k(p)((z))^*\simeq(\hat{\mathcal{O}}_p)_0$, by considering the artinian local ring $A=k[\epsilon]/\epsilon^3$, one has that

$$\begin{aligned}(1-\epsilon f,1-\epsilon g)_{k(p)}&=N_{k(p)/k}(1-\epsilon^2\operatorname{res}_p(gdf))\\&=1-\epsilon^2\operatorname{Tr}_{k(p)/k}(\operatorname{res}_p(gdf)).\end{aligned}$$

Hence, from the above reciprocity law, we can recover the expression of the generalized residue theorem over a perfect field:

$$\sum_{p\in C}\operatorname{Tr}_{k(p)/k}(\operatorname{res}_p(gdf))=0.$$

Remark 4.5 (Generalized Witt Reciprocity Law): With the notations of [1] (Section 4.3), if k is a perfect field and $k\hookrightarrow k(s)$ is a finite extension, taking the artinian local ring $A=k[\epsilon]/\epsilon^{N+1}$ and bearing in mind that $\mathbb{W}_{\leq N}(k)\simeq\mathbb{G}_m^\bullet(\operatorname{Spec}A)$ and $\mathbb{W}_{\leq N}(k(s)((t)))\simeq\Gamma_{k(s)}{}^\bullet(\operatorname{Spec}A)$, we can define the pairing

$$\operatorname{res}^{\mathbf{W}}_{\leq N}(\cdot,\cdot)_{k(s)}\colon\widetilde{\Gamma_{k(s)}}^\bullet(\operatorname{Spec}k)\times\mathbb{W}_{\leq N}(k(s)((t)))\to\mathbb{W}_{\leq N}(k)$$

by the rule

$$\Big(f,\prod_{i=1}^N(1-x_i\epsilon^i)\Big)_{k(s)}\equiv\prod_{i=1}^N(1-\epsilon^i(\operatorname{res}^{\mathbf{W}}_{\leq N}(f,x)_{k(s)})_i)\bmod(\epsilon^{N+1})$$

where $(\cdot,\cdot)_{k(s)}$ is the generalization of the Contou-Carrère symbol associated with the separable extension $k\hookrightarrow k(s)$.

Thus, if C satisfies the hypothesis of the theorem, we deduce from the above reciprocity law that

$$\sum_{p\in C} \operatorname{res}^{\mathbb{W}}_{\leq N}(f_p, x_p)_{k(p)} = 0$$

for all $f \in \Sigma_C^*$ and $x = (x_i)_{i=1}^N \in \mathbb{W}_{\leq N}(\Sigma_C)$, and where the addition is to be performed in the group $\mathbb{W}_{\leq N}(k)$. When k is an algebraically closed field of characteristic p and $N = p^{n-1}$, according to the considerations made in [1], it is possible to recover the Witt reciprocity law [16] from this expression. Hence, the last formula is somehow a generalization of the Witt Reciprocity Law.

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